§2 习题课

1. 设 $w=f(z)$ 在区域 D 上单叶解析，则其反函数 $z=f^{-1}(w)$ 在 $G=f(D)$ 上也解析，且

$$\frac{df^{-1}(w)}{dw}=\frac{1}{f'(z)}$$

Pf. $\forall z_0\in D$, $f(z_0)=w_0\in G$，则 $\left.\frac{df^{-1}(w)}{dw}\right|_{w_0}=\lim\limits_{w\to w_0}\frac{f^{-1}(w)-f^{-1}(w_0)}{w-w_0}$

$$=\lim_{w\to w_0}\frac{z-z_0}{w-w_0}=\frac{1}{\lim\limits_{z\to z_0}\frac{w-w_0}{z-z_0}}=\frac{1}{f'(z_0)}$$

Rem. (1) 若 D 是区域，其像集 $G=f(D)$ 也是区域（f 解析）

(2) 若 f 在 D 上单叶解析，则 $f'(z)\neq 0$.

2. 设 $w=f(z)$ 在有界区域 D 上单叶解析，则 D 在变换 $w=f(z)$ 下的像 $G=f(D)$ 的面积

$$A=\iint_D |f'(z)|^2 d\sigma$$

Pf. 记 $w=f(z)=u(x,y)+iv(x,y)$，则满足 $u_x=v_y$, $u_y=-v_x$

G 的面积 $A=\iint\limits_G du\,dv \xlongequal[v=v(x,y)]{u=u(x,y)} \iint\limits_D |J|\,dx\,dy=\iint\limits_D (u_x^2+v_x^2)\,d\sigma$

$$=\iint_D |f'(z)|^2 d\sigma$$

3. 设 $w=f(z)$ 在上半平面上解析 $\Longleftrightarrow$ $w=\overline{f(\bar z)}$ 在下半平面上解析

Pf. $\forall z_0$, $\operatorname{Im} z_0<0$, $f'(\bar z_0)$ 存在

$$\left.\frac{dw}{dz}\right|_{z_0}=\lim_{z\to z_0}\frac{\overline{f(\bar z)}-\overline{f(\bar z_0)}}{z-z_0}=\overline{\lim_{\bar z\to \bar z_0}\frac{f(\bar z)-f(\bar z_0)}{\bar z-\bar z_0}}=\overline{f'(\bar z_0)}$$

同理反向可证.

Rem. 若 $w=f(z)$ 在上半平面上解析且 $\operatorname{Im} f(x)=0$，则 $f(z)$ 可延拓为全平面的解析函数

事实上有：$F(z)=\begin{cases} f(z), & \mathrm{Im}\, z \geq 0 \\ \overline{f(\bar{z})}, & \mathrm{Im}\, z < 0 \end{cases}$

4. 已知 $u=x^2+xy-y^2$，求 v 使 $f(z)=u+iv$ 解析.

Sol. $\frac{\partial u}{\partial x}=2x+y=\frac{\partial v}{\partial y}$，$v=2xy+\frac{1}{2}y^2+C(x)$

$\frac{\partial v}{\partial x}=2y+C'(x)=-\frac{\partial u}{\partial y}=-x+2y$，$C'(x)=-x$，$C(x)=-\frac{1}{2}x^2+C$

$v=2xy+\frac{1}{2}y^2-\frac{1}{2}x^2+C$

$f(z)=x^2+xy-y^2+i\left(2xy+\frac{1}{2}y^2-\frac{1}{2}x^2+C\right)$，$C$ 为实常数

5. 设 $u(x,y)$ 在 D 上~~有二阶连续偏导数~~是调和函数，复变函数 $z=f(\zeta)=x+iy=x(\xi,\eta)+iy(\xi,\eta)$ 在 ζ 平面上解析，则复合函数 $u(x(\xi,\eta),y(\xi,\eta))$ 在 G 上也是调和函数.

Pf. 根据题意，$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\equiv 0$，$\frac{\partial x}{\partial \xi}=\frac{\partial y}{\partial \eta}$，$\frac{\partial x}{\partial \eta}=-\frac{\partial y}{\partial \xi}$

$$\frac{\partial^2 x}{\partial \xi^2}+\frac{\partial^2 x}{\partial \eta^2}=0,\quad \frac{\partial^2 y}{\partial \xi^2}+\frac{\partial^2 y}{\partial \eta^2}=0$$

$$\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial \xi^2}=\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial^2 u}{\partial x\partial y}\cdot\frac{\partial y}{\partial \xi}\right]\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial x}\cdot\frac{\partial^2 x}{\partial \xi^2}+$$

$$\left[\frac{\partial^2 u}{\partial y\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \xi}\right]\frac{\partial y}{\partial \xi}+\frac{\partial u}{\partial y}\cdot\frac{\partial^2 y}{\partial \xi^2}$$

同理 $\frac{\partial^2 u}{\partial \eta^2}=\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial \eta}\right]\frac{\partial x}{\partial \eta}+\frac{\partial u}{\partial x}\cdot\frac{\partial^2 x}{\partial \eta^2}+\left[\frac{\partial^2 u}{\partial y\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \eta}\right]\frac{\partial y}{\partial \eta}+\frac{\partial u}{\partial y}\cdot\frac{\partial^2 y}{\partial \eta^2}$

两式相加，

$$\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\cdot\left[\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial y}{\partial \xi}\right)^2\right]=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\cdot|f'(z)|^2=0$$

Cor. $u(z)$ 调和，$z=f(\zeta)$ 解析，则 $u(f(\zeta))$ 也调和.

6. (1) 证明多值函数 $w=\sqrt[3]{z(1-z)}$ 的支点为：0, 1, ∞

(2) 用正实轴割破 z 平面后，$w=\sqrt[3]{z(1-z)}$ 分出三个分支函数 $w_k=\left(\sqrt[3]{z(1-z)}\right)_k$，已知某一支 w_k 满足 $w_k(-1)<0$，求 $w_k(-i)$.

Sol. (1) 先证 $z=0$ 为支点：取绕 $z=0$ 的封闭曲线 C（动点 z 绕沿 C 逆时针一周回到原位置），这里 C 不含 $z=1$，则

$\Delta_C \arg z = 2\pi$，$\Delta_C \arg(1-z)=0$，$\Delta_C \arg\sqrt[3]{z(1-z)} = \frac{2\pi+0}{3} = \frac{2\pi}{3}$，

C 的像曲线没有回到原位置，故 $z=0$ 为支点，

同理 $z=1$ 也是支点，

考虑 $z=\infty$ 顺时针（C 内含 $z=0$, $z=1$），$\Delta_C \arg\sqrt[3]{z(1-z)} = \frac{-2\pi-2\pi}{3} = -\frac{4\pi}{3}$

因此 $z=\infty$ 也是支点，

(2) 设 $z=r_1e^{i\theta_1}$，$1-z=r_2e^{i\theta_2}$

$w_k=\sqrt[3]{z(1-z)}=\sqrt[3]{r_1r_2}\,e^{\frac{\theta_1+\theta_2+2k\pi}{3}i}$，$k=0,1,2$

若 $w_k(-1)=\sqrt[3]{2}\,e^{\frac{\pi+0+2k\pi}{3}i}<0$，$k=1$，

则 $w_1(-i)=\sqrt[6]{2}\,e^{\frac{\frac{3\pi}{2}+\frac{\pi}{4}+2\pi}{3}i}=\sqrt[6]{2}\,e^{\frac{5}{4}\pi i}$

7. 已知 $w=\sqrt{z(z-1)}$ 去掉正实轴上线段 $[0,1]$ 分出两个分支函数 $w_k=\left(\sqrt{z(z-1)}\right)_k$，已知某一支 w_k 满足 $w_k(-1)>0$，求 $w_k(-i)$.

Sol. $z=0$, $z=1$, ~~z=∞~~ 为支点，$z=\infty$ 非支点，

设 $z=r_1e^{i\theta_1}$，$z-1=r_2e^{i\theta_2}$

$w_k=\sqrt{r_1r_2}\,e^{\frac{\theta_1+\theta_2+2k\pi}{2}i}$，$k=0,1$

若 $w_k(-1)=\sqrt{2}\,e^{\frac{\pi+\pi+2k\pi}{2}i}>0$，$k=1$

则 $w_1(-i)=\sqrt[4]{2}\,e^{\frac{\frac{3\pi}{2}+\frac{5\pi}{4}+2\pi}{2}i}=\sqrt[4]{2}\,e^{\frac{3\pi}{8}}$